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The Genesis of a Manifold with Boundary: A Model of the Archytas Paradox

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Abstract: A geometrical model of cosmos is proposed whereby Archytas' argument against the existence of an "edge of the heaven" does not work. The model, based on the solution to a previous paradox concerning non-rectifiable lines, demonstrates the possibility of a finite and bounded Euclidean space and reveals the purely geometrical origin of certain forces present therein.

Keywords:
Archytas Paradox; Archytas force; bounded Euclidean space; manifold with boundary; non-rectifiable line; spatial edge

1. The Archytas Paradox

It is widely acknowledged that Archytas attacked the conception of a finite universe with a thought experiment that has influenced cosmologists for two thousand years, and is one of the first thought experiments in antiquity. [1] Let us consider the presentation that Simplicius makes of Archytas' reasoning through Eudemus:

[1] See Roy A. Sorensen, **Thought Experiments**, Oxford University Press, Oxford 1999; John D. Norton, "Why Thought Experiments do not Transcend Empiricism", in: Christopher Hitchcock (ed.), **Contemporary Debates in Philosophy of Science**, Blackwell Publishing, Malden 2004, pp. 44–66; Carl Huffman, "Archytas", in: Edward N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy,* Summer 2018 Edition, https://tiny.pl/rm64s9jb [14.04.2025].

> But Archytas [as Eudemus says] used to propound the argument in this way: "If I arrived at the outermost edge of the heaven [that is to say at the fixed heaven], could I extend my hand or staff into what is outside or not? It would be paradoxical not to be able to extend it. But if I extend it, what is outside will be either body or place. It doesn't matter which, as we will learn. So then he will always go forward in the same fashion [...] and thus there would be unlimited body and space". [2]

The conclusion is that space is unbounded. Here Archytas' argument, in principle, runs counter to the possibility of a bounded cosmos ("edge of the heaven"), regardless of whether such a cosmos is finite or not. If one then maintains that a finite cosmos implies a cosmos with a boundary (edge), then Archytas' argument leads to the denial of its finitude, but such finitude is not essential to the argument. Focusing on this, what shall be referred to as the Archytas paradox is clearly detailed: if the cosmos has an edge, it would be paradoxical if I were unable to stretch my hand beyond it. In more succinct terms: How is it possible that I cannot stretch my hand beyond an edge of space? It would be paradoxical if I could not do so. In response to the fact that, by definition, there is nothing (and no space) beyond an edge, the paradox is approached once again in the following terms: How is it possible that there is a spatial edge (boundary) in the above sense? The existence of such an edge, understood as a purely spatial edge, would be paradoxical. Any force made manifest by a spatial edge in this sense will subsequently be called an Archytas force. This idea must be clearly distinguished from that of an edge (boundary) *in* space (and, therefore, not purely spatial). Such an edge could be a rigid barrier running along a part of it (or some arbitrarily intense force field) that blocks any kind of breach. In contrast, a purely spatial edge (boundary) would imply, as Huggett states, that every attempt to pass the edge would fail, but not because anything in particular stops you. [3] After a digression on some non-rectifiable lines in the following section, a model of the Archytas paradox will be discussed in the

[2] Monte Ransome Johnson, "Sources for the Philosophy of Archytas", *Ancient Philosophy* 2008, Vol. 28, No. 1, p. 186 [173–199], https://doi.org/10.5840/ancientphil20082819.

[3] See Nick Huggett, **Everywhere and Everywhen: Adventures in Physics and Philosophy**, Oxford University Press, Oxford 2010.

third section in the precise sense suggested by Huggett. [4] The two concluding sections comprise a defence of the paradox's relevance and my model of it.

2. Interlude

Suppose that a runner C moves on a continuous, rectilinear piecewise trajectory, Ω, such as that drawn on plane Π in Figure 1. It consists of: a) three rectilinear fragments: to the left of L_1 (where it moves away to infinite $-\infty$), between L and R, and to the right of R_1 (where it moves away to infinite $+\infty$); b) two decreasing saw-toothed fragments: between L_1 and L to the left and between R and R_1 to the right. Initially C is on the LR line and moves, so to speak, to the right at unit speed. For all $n \geq 1$, triangles $L_{2n+1}L_{2n}L_{2n-1}$ are isosceles (equal sides $L_{2n+1}L_{2n}=L_{2n}L_{2n-1}$), and respectively congruent with $R_{2n+1}R_{2n}R_{2n-1}$ (equal sides $R_{2n+1}R_{2n}=R_{2n}R_{2n-1}$). Let us assume that their bases are $L_{2n+1}L_{2n-1}=R_{2n+1}R_{2n-1}=1/2^n$ and their heights $h(L_{2n+1}L_{2n}L_{2n-1})=h(R_{2n+1}R_{2n}R_{2n-1})=1/n^2$. Despite the infinite crowding of increasingly smaller triangular edges immediately to the right of R, the runner can easily pass this point and travel along the toothed fragment of the trajectory to the right. Indeed, since any side is smaller than the sum of the other two in every triangle, in triangles $R_{2n+1}R_{2n}R_{2n-1}$, it follows that $R_{2n+1}R_{2n}=R_{2n}R_{2n-1}<(1/n^2)+(1/2^{n+1})$. This implies that the total length of the rightmost toothed trajectory is less than the infinite sum $\Sigma\{(1/n^2)+(1/2^{n+1})\}$ Yet this series is convergent. Thus, moving at unit speed, the runner will travel along it completely in less than $\Sigma\{(1/n^2)+(1/2^{n+1})\}$ counting from moment $t=0$, when they passed R.

Let us alternatively assume that C departs from point R at $t=0$ with the intention of moving on the right-hand toothed trajectory at a non-constant speed, but equal to $v=2t$. It is a purely technical question determining at which point on the trajectory they will be at any given time $t^*>0$. Furthermore, as their acceleration in the direction of motion is $a=2$, it follows that, in order to carry out their journey, the runner (who is assumed to be of unit mass) is boosted by a constant force of magnitude 2 in the direction of motion. Either of the above two situations shall be referred to as the non-paradoxical case.

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[4] Here I interpret the term "paradox" in the broad sense. Not as a contradiction, but as something highly counter-intuitive; in the same sense that we speak, for example, of the Banach-Tarski paradox.

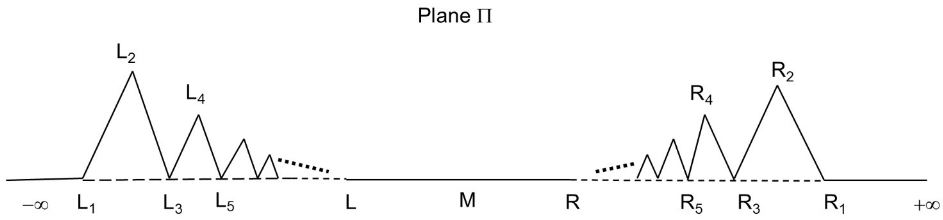

Figure 1. A continuous, rectilinear piecewise trajectory

Things will now be altered by making just one change to the whole discussion. The heights of triangles $L_{2n+1}L_{2n}L_{2n-1}$ and $R_{2n+1}R_{2n}R_{2n-1}$ will no longer have a value of $1/n^2$ but $1/n.h(L_{2n+1}L_{2n}L_{2n-1})=h(R_{2n+1}R_{2n}R_{2n-1})=1/n$. As in any right triangle the hypotenuse is longer than any of the legs, in triangles $R_{2n+1}R_{2n}R_{2n-1}$ (consisting of two right triangles with a common leg) it follows that $R_{2n+1}R_{2n} = R_{2n}R_{2n-1}>1/n$. This implies that the total length of the toothed trajectory on the right is greater than the infinite sum $\Sigma 1/n$. Yet this series is divergent. Meaning that, moving at unit speed, the runner will never be able to travel along it (in a finite time). Indeed, for the same reason, any sub-trajectory of the above that goes from R to $R_m(m\geq 1)$ also has an infinite length, meaning that it cannot be followed either. The conclusion is that a runner approaching R from the left at unit speed cannot pass this point if they must always travel along the trajectory in Figure 1. What prevents them from doing so? If C departs from point R at $t=0$ with the intention of travelling along the right toothed trajectory at a non-constant speed but equal to $v=2t$, they will unable to do so. If, as in the non-paradoxical case, they move under a constant force of magnitude 2, an equal and opposite force must arise to cancel it out, preventing motion. Who exerts this force? Either of the above two situations shall be referred to as the paradoxical case.

The cause of these strange forces preventing motion cannot be the infinite crowding of increasingly smaller triangular edges immediately to the right of R, for this also occurred in the non-paradoxical case. The only change in one case compared to the other was that the triangle heights went from taking infinitely decreasing values $1/n^2$ to taking infinitely decreasing values $1/n$. A solution to this enigma shall be put forward by idealizing the way in which runner C operates.

Let us assume that C is a particle (ideally punctual) subjected to the constraint of having to move without ever abandoning the Ω trajectory in Figure 1. A particle moving along this trajectory is generally subject to constraining forces. These forces typically act at the points where the particle changes direction abruptly in order to maintain the trajectory to which it is constrained. This occurs both in the paradoxical and non-paradoxical case. The difference lies in the fact that, in the first case, an additional constraining force, F, necessarily arises, preventing the particle from progressing on Ω further to the right of R. It is impeded because there are no points on Ω located to the right of R that are at a finite distance from R measured along trajectory Ω that C is forced to follow. The only points on Ω that are within a finite distance of R (measured along trajectory Ω that C is forced to follow) are points to the left of R. This additional force F exerted on C is yet another manifestation of the forces arising from the external constraints preventing their motion outside the Ω trajectory in Figure 1, which are exerted on C by the complex rigid structure along which it is forced to move. If the particle approaches from M to R at unit speed (in the so-called paradoxical case), an instantaneous constraining force at R then causes it to bounce backwards. To summarize this paradoxical case, if C reaches R_n from R_{n+1} at unit speed, there are points on Ω at finite distance from R_n (situated to the right of R_n) which C can then occupy (the constraints allow C to continue motion to the right of R_n). However, if C reaches R from M, the only points on Ω at finite distance from R which C can then occupy are to the left of R. Therefore, the constraints in this case involve bouncing backwards.

The analysis carried out in this section will help to construct an interesting model of the Archytas paradox in the section that follows. It concerns a space where the paradoxical forces of Archytas (surprisingly) have purely geometrical origins.

3. A Model of the Archytas Paradox

Let us now move on to the Archytas' paradox model.

3.1 One dimensional

Let us assume that the Ω line in Figure 1 is Lineland, an infinite one-dimensional world inhabited by one-dimensional beings (segments of finite length). Lineland exists on infinite plane Π, Flatland, which is an infinite two-dimensional world. For reasons seen later, this one-dimensional infinite world shall be named (e)Lineland (for "extrinsic Lineland"). The triangle heights will be assumed to have a value of $1/n(h(L_{2n+1}L_{2n}L_{2n-1})=h(R_{2n+1}R_{2n}R_{2n-1})=1/n, n \geq 1)$. The inhabitants of (e)Lineland are constrained to moving along Ω. This constraint acts on them in several ways. For instance, if they move from R_3 to R_1, it will act on R_2 to "bend" them as required. Similarly, a (e)Linelander departing from *M* moving at a constant speed to the right will be unable to pass point *R*. A (constraining) force will act there which prevents them, as it did on the particle moving on the lattice in the previous section. There is a natural metric defined on Ω: the distance between any two points *p* and *q* on Ω, $d(p,q)$, is the length of the Ω section between these points. This length, which is fully specified, is a definite known function of two variables $L[\ ,\ ]$. In other words, $d(\ ,\ )=L[\ ,\ ]$. For example, it is known that $d(R_{2n+1},R_{2n})=L[R_{2n+1},R_{2n}]=\sqrt{[(1/n^2)+(1/2^{2n+2})]}$. There is also a natural parametrization for (e)Lineland that enables its points based on the values of a parameter to be unequivocally identified. In order to define this, it should be noted that Ω rests on an infinite horizontal line, $\Re$, which will be identified by the real number line. One, and only one, point $P(x)$ on $\Re$ corresponds to each real number *x* (and *vice versa*). In addition, one, and only one, point $Q(P(x))\equiv(Q\circ P)(x)$ on Ω corresponds to each point $P(x)$ on $\Re$ (and *vice versa*). This latter correspondence is clear and natural: $Q(P(x))$ is the point on Ω whose perpendicular projection on $\Re$ is precisely point $P(x)$ on $\Re$. Clearly, when $Q(P(x))$ is on $\Re$, $Q(P(x))$ therefore coincides with $P(x)$. Given the symmetry in Figure 1, it is also natural that, with *M* being the midpoint between *L* and *R*, it can be affirmed that $M=P(0)=Q(P(0))$. *So*, $(Q\circ P)^{-1}(L)=P^{-1}(L)=-P^{-1}(R)=-(Q\circ P)^{-1}(R)$. It is also clear, for example, that $(Q\circ P)^{-1}(R_1)=(Q\circ P)^{-1}(R)+1/2+1/4+1/8+...=(Q\circ P)^{-1}(R)+1$, and that $(Q\circ P)^{-1}(R_2)=(Q\circ P)^{-1}(R_1)-1/4$, ... etc. Based on this parametrization, the [5] (e)Lineland metric can now be rewritten as follows. For any real numbers x_1, x_2, $d(Q(P(x_1)), Q(P(x_2)))=L[Q(P(x_1)),Q(P(x_2))]\equiv f(x_1,x_2)$. In other words: $d(Q(P(\ )),Q(P(\ )))=L[Q(P(\ )),Q(P(\ ))]$ with $f(\ ,\ )\equiv L[Q(P(\ )),Q(P(\ ))]$. (e)Lineland consists of three "worlds" that have no communication with one another (see Figure 2): (e)Lineland$^-$ (points on Ω to the left of *L*), (e)Lineland^0(points on Ω between *L*

[5] Strictly speaking this is not a metric, since the distance between certain points in (e)Lineland is not defined (it is "infinite"), for example, between R and R_1. However, there is no need for concern about that here. However, see section 3.3 below.

and *R*, including both) and (e)Lineland$^+$ (points on Ω to the right of *R*). The distance between pairs of points belonging to different "worlds" is infinite. Also, (e)Lineland^0 is a finite and bounded world. As seen above, the inhabitants (or particles) in (e)Lineland^0 that reach *R* or *L* cannot pass them. [6] This is due to the presence of forces arising from the constraints acting from plane Π in Figure 2 (Flatland), where (e)Lineland exists.

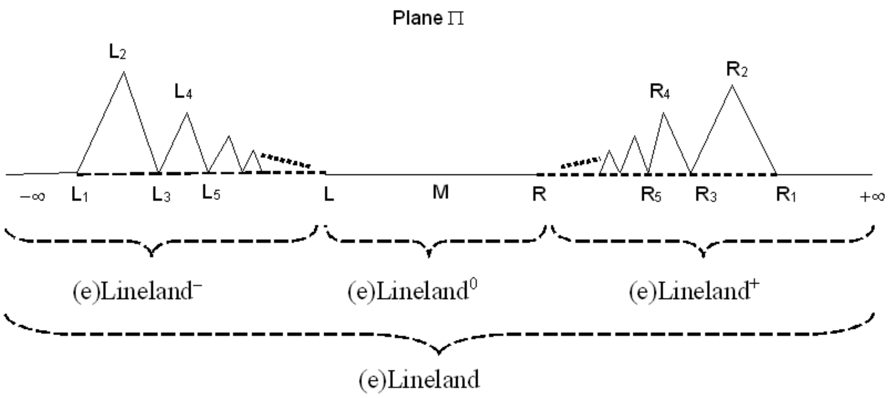

Figure 2. (e)Lineland

Nevertheless, let us alternatively assume that the original Lineland (with all the intrinsic properties of (e)Lineland) is not actually embedded in any higher dimensional space. All the earlier discussion is still valid so long as it is kept in mind that now *Q*( ) lacks visual interpretation, and is just a purely formal application. For reasons that are now clear, this one-dimensional infinite world will now be named (i)Lineland (for "intrinsic Lineland"). The points in (i)Lineland are not the $Q(P(x))$ but points $P(x)$ on $\Re$. It is therefore a one-dimensional space that is trivially parametrized by real numbers. Furthermore, since the intrinsic properties of (e)Lineland are preserved (the metric being one of them), this space must be provided with a metric determined by $d(P(x_1),P(x_2))=f(x_1,x_2)$, where $f(x_1,x_2)$ is exactly the same function as before; $f(x_1,x_2)=L[Q(P(x_1)),Q(P(x_2))]$. However, *Q( )* does not represent anything physical and is simply a formal

[6] The inhabitants of (e)Lineland$^-$ or (e)Lineland$^+$ can neither reach R nor L because the distance separating them from these points is infinite. (e)Lineland$^-$ and (e)Lineland$^+$ are infinite worlds.

resource for calculating distances.[7] (i)Lineland can be imagined as an infinite straight line in both directions, and provided with a metric which is non-uniform. This non-uniformity arises from the fact that there are points on it such as L and R (with coordinates $P^{-1}(L)$ and $P^{-1}(R)$) whose distance is $d(L, R)=|P^{-1}(L)-P^{-1}(R)|$ and points such as R_1 and R_3 ((with coordinates $P^{-1}(R_1)$ and $P^{-1}(R_3)$) whose distance is $d(R_1, R_3)=2\sqrt{(1+1/16)}>|P^{-1}(R_1)-P^{-1}(R_3)=1/2$. Let us consider (i)Lineland^0, and an astronomer *A* who lives there. It is easy to see that the cosmos model for *A* is a finite and bounded universe, the simplest exemplification of the Archytas paradox. A can move around in (i)Lineland^0 and the distance between any two points therein, *S* and *T*, is the natural distance $d(S,T)=|P^{-1}(S)-P^{-1}(T)|$.[8] However, this world has two edges: *R* and *L*. A can reach R but not pass it. There is no object preventing this (nor any physical constraint); they simply cannot occupy a region on the other side because there is no "other side" of *R*. What does no "other side" of *R* (or *L*) mean? A's cosmos is the set of points at a finite distance from any point in (i)Lineland^0, and the only points in (i)Lineland within finite distance of a point in (i)Lineland^0 are the actual points in (i)Lineland^0. So, *A* is unable to extend a one-meter ruler to the other side of *R* because there are no points in (i)Lineland within a finite distance of *R* which are not points belonging to (i)Lineland^0. There are no points on the other side of *R* within a finite distance of *R*: in this precise sense, there is no "other side" of *R*.[9] For *A* to be able to slide his one-meter ruler through *R* would require points that could be occupied on the other side of *R*, and there are no such points. *R* is a purely spatial edge, not an edge *in* space (spatial points are not in space, rather, they constitute it). What stops the astronomer *A* from advancing is not something that is in *R* (or in *L*). However, as far as being unable to advance is concerned, we can speak of a force preventing them (as in the paradoxical case in the section above) which one tempts to name the "Archytas force". This Archytas force is, nonetheless, of exclusively geometrical origin. In

[7] Below we shall see that, now in two dimensions, Q( ) plays the role of making the geometry of part of the space non-Euclidean.

[8] Unlike (i)Lineland, (i)Lineland^0 is a metric space in the strict sense.

[9] But obviously there are points "on the other side" of edge R that are at an infinite distance from R (such as R_n). This does not in any way detract from the model's role as a representation of the Archytas paradox. Rather, it demonstrates that this paradox (the Archytas staff problem) needs to be distinguished from the problem of whether there actually is space beyond the edge. Historically, they have tended to be considered equivalent, however, now we see they are not.

this precise sense, the model of (i)Lineland studied constitutes a realization of the Archytas paradox.

3.2 Two dimensional

An even more interesting model can be obtained in two steps:

1) First of all, by going beyond (e)Lineland to (e)Flatland, (e)Flatland can be generated by freely rotating (e)Lineland around an axis located on plane *Π*, and perpendicular to line *LR* passing through point *M*. The (e)Flatland thus generated exists (is embedded) in a higher dimensional (three-dimensional) space. It is really a surface (Ω^2) which is partly flat and partly deformed by circular craters which are all concentric with point *M*. There is a natural metric defined on Ω^2. The distance between any two points p and q on Ω^2, $d(p,q)$, is: a) the length of the shortest path from p to q on Ω^2; b) infinite, if all paths from p to q on Ω^2 are of infinite length. As is evident (see Figure 3), (e)Flatland consists of two "worlds" that have no communication with one another: (e)Flatland$^{\pm}$ (points on Ω^2 outside the LR-diameter circle), (e)Flatland^0 (points on Ω^2 inside the LR-diameter circle or on its C_{LR} circumference). The distance between pairs of points belonging to different "worlds" is infinite. Also, (e)Flatland^0 is a finite and bounded world. Analogous to what occurred in (e)Lineland^0, the inhabitants (or particles) of (e)Flatland^0 that reach C_{LR} cannot pass it. This is due to the presence of forces arising from constraints acting on the three-dimensional space where Ω^2 exists.

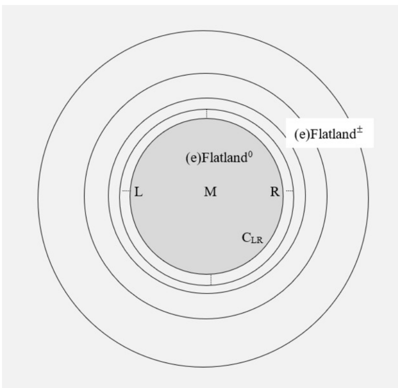

Figure 3. (e)Flatland

2) Second, by assuming that the original Flatland (with all the intrinsic properties of (e)Flatland) is not actually embedded in a higher dimensional space. For the same reasons as in the case of Lineland, this infinite two-dimensional world will be named (i)Flatland (for "intrinsic Flatland"). The points in (i)Flatland are points $P(x,y)$ on $\Re x\Re \equiv \Re^2$ (obtained from (e)Flatland by perpendicular projection on plane $\Re^2$, on which it rests). It is therefore a two-dimensional space that is trivially parametrized by all the pairs of real numbers. Furthermore, since the intrinsic properties of (e)Flatland are preserved (the metric being one of them), (i)Flatland can be imagined as an infinite plane in all directions and provided with a metric that is non-uniform, and that naturally generalizes (i)Flatland's metric.

Let us now consider (i)Flatland^0 and astronomer *A*, who lives there. A's cosmos model is also a finite and bounded universe; an example of the Archytas paradox. A can move freely in (i)Flatland^0, which (despite its finiteness and boundaries) has Euclidean geometry.[10] However, *A*'s world has an edge: circumference C_{LR}. A can reach it but not pass it since there is no "outside" of C_{LR}. The only points in (i)Flatland at a finite distance from circumference C_{LR}, or from its internal points, are the actual points on C_{LR}, or its actual internal points. In this precise sense, there is no "outside" of C_{LR}. C_{LR} is a purely spatial edge, not an edge IN space. The Archytas force (of geometric origin) now acts on the circular edge of a finite, bounded, two-dimensional Euclidean world.

3.3 Three dimensional

On the basis of the two-dimensional case analyzed, there is a beautiful and simple procedure to find a model of the Archytas paradox in a finite, bounded and three-dimensional Euclidean world. It begins with the ordinary *XY* infinite plane but provided with the (i)Flatland metric. The origin of coordinates *O*(0,0) is then made to coincide with point *M* in (i)Flatland. If the *XY* plane is rotated around the *O* origin in all possible ways (i.e. at any angle and around any axis of rotation passing through *O*), all of the three-dimensional space is generated. In order to

[10] As is evident, the geometry of (i)Flatland$^{\pm}$ is not Euclidean. (i)Flatland$^{\pm}$ does not even have the same topology as (i)Flatland^0 since the latter is simply connected while the former is not (it has a "hole").

obtain (i)Spaceland, this three-dimensional space is given a simple metric, defined as follows. Taking any two points *P* and *Q* in the three-dimensional space, the *XY* plane ((i)Flatland) is rotated with a fixed point *O*(0,0) until a plane is obtained (which shall be called *OPQ*) that also passes through *P* and *Q*. This plane exists (three points, in this case *O,P* and *Q*, always determine at least one plane). Since the *OPQ* plane has (i)Flatland's metric, the distance between *P* and *Q* on this plane is, by definition, the distance between points *P* and *Q* in the three-dimensional space. This space will be named (i)Spaceland (see Figure 4). It is now clear that (i)Spaceland^0 is a closed ball of diameter *LR* and surface S_{LR} (it consists of all points at finite or zero distance from origin *M*), and that (i)Spaceland$^\pm$ is the outside of this ball.

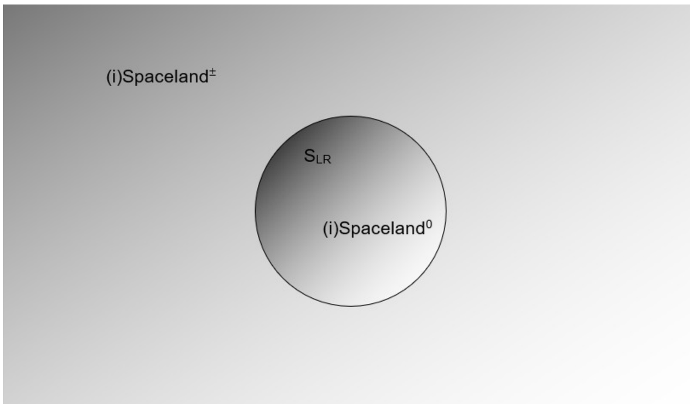

Figure 4. (i)Spaceland

The Archytas force acts on edge S_{LR} of (i)Spaceland^0; a finite, bounded and three-dimensional Euclidean world. Contrary to widely-held intuition, a three-dimensional Euclidean world neither has to be infinite [11] nor unbounded. This force is not exerted by "something", rather it is a consequence of a constraint of a purely geometric nature linked to the actual existence of edge S_{LR}. In this regard, it is similar to the "forces" encountered in general relativity, related to the existence

[11] Consider, for example: "Archytas' argument would still amount to a sufficiently reasonable proof that if our space is Euclidean then it has to be infinite." George N. Schlesinger, "The Power of Thought Experiments", *Foundations of Physics* 1996, Vol. 26, No. 4, p. 478 [467–482], https://doi.org/10.1007/BF02071216.

of timelike geodesics in spacetime geometry. Timelike geodesics prescribe what a test particle can do. S_{LR} prescribes what it cannot do. In any event, the Archytas force prevents the presence of the edge of space in (i)Spaceland^0 from entering into contradiction with Newton's first law, as Le Poidevin seems to think. [12] There is an important way in which S_{LR} is analogous to the Aristotelian sphere of stars in that (i)Spaceland provides, quite literally, a model where Aristotle's cosmos can find its place. Consider, for instance, what is stated in **Physics**: "but the heaven [...] is not anywhere as a whole, nor in any place, if at least, as we must suppose, no body contains it". [13] This idea is consistent with my model in that (i)Spaceland^0 is nowhere to be found in (i)Spaceland, and this is for the simple reason that, strictly speaking, there is no defined metric for the entirety of (i)Spaceland (see note 2). For the same reason, as for Aristotle "there can be no other *cosmoi* located outside of our own", [14] nor is there any other space *located* outside (i)Spaceland^0 ((i)Spaceland$^{\pm}$ is at an infinite distance from it). [15]

4. The Archytas paradox in context

The Archytas paradox has not historically gained the same kind of relevance as several of the classical paradoxes dating back to antiquity, such as the Zeno (or liar) paradoxes. Nonetheless, failing to acknowledge that there is a rich context of ideas surrounding this paradox in the history of philosophy would be a mistake. Vilenkin recalls that Archytas was likely inspired by Anaximander, who, in order to clarify the idea of unboundedness, stated that "wherever the warrior stands he can extend his spear farther". [16] The paradox of Archytas appears centuries later

[12] See Robin Le Poidevin, **Travels in Four Dimensions: The Enigmas of Space and Time**, Oxford University Press, Oxford 2003.

[13] Aristotle, **Physics**, Book IV, 212b5–10, in: Richard McKeon (ed.), **The Basic Works of Aristotle**, Random House, New York 1941, pp. 269–302.

[14] In the words of Belot (Gordon Belot, **Geometric Possibility**, Oxford University Press, Oxford 2011, p. 159).

[15] Belot also states that, in Archytas' argument, the existence of the void functions only as a sort of place-holder for possible deformations or expansions of the cosmos. Given what I have just said, neither are there place-holders in (i)Spaceland for possible deformations or expansions of (i)Spaceland^0.

[16] Naum Ya Vilenkin, **In Search of Infinity**, Birkhäuser, Boston 1995, p. 3.

in Lucretius' **De rerum natura**. Indeed, Rucker goes so far as to assert that "Lucretius first gave the classic argument for the unboundedness of space". [17] And the Middle Ages provide the first model of a finite and unbounded universe in Dante's **Divine Comedy**. Rovelli provides some very interesting details in this regard which focus on the Canto XXVIII of **Paradise**, [18] and are based on Petersen's detailed analysis. [19] He considers that this work by the Italian poet presents a solution to the Archytas problem which pre-empts Einstein's celebrated "Kosmologische Betrachtungen" of 1917. Nor should the relationship between the Archytas argument and the Kantian antinomies be overlooked from a historical perspective. Approvingly quoting Martin, Priest makes the following statement: "Kant saw quite clearly that the antinomies rest on this antithesis between making a conclusion and going beyond the conclusion. In principle, this had already been seen by Archytas, when he wanted to go to the end of the world and stretch out his arm". [20] Weber also recalls the Archytas argument (in close connection with a thought of Wittgenstein's in the **Tractatus**) and seems to want to relocate it within the framework of modern *dialetheism* ("There *are* thoughts we cannot think — and we are thinking one of them *right now*" [21]). Finally, Cini and Fano evoke the Archytas paradox to explain why (in the context of differential spacetime geometry) the bulk of current research focuses on manifolds without boundary. [22]

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[17] Rudy Rucker, **Infinity and the Mind: The Science and Philosophy of the Infinite**, Princeton University Press, Princeton 1995, p. 15.

[18] See Carlo Rovelli, "Some Considerations on Infinity in Physics", in: Michael Heller and Hugh W. Woodin, **Infinity: New Research Frontiers**, Cambridge University Press, Cambridge 2011, pp. 167–175.

[19] See Mark A. Petersen, "Dante and the 3-sphere", *American Journal of Physics* 1979, Vol. 47, No. 12, pp. 1031–1035, https://doi.org/10.1119/1.11968.

[20] Graham Priest, **Beyond the Limits of Thought**, Cambridge University Press, Cambridge 1995, p. 134; Gottfried Martin, **Kant's Metaphysics and Theory of Science**, Manchester University Press, Manchester 1955.

[21] Zach Weber, "At the Limits of Thought", in: Can Baskent and Thomas Macaulay Ferguson (eds.), **Graham Priest on Dialetheism and Paraconsistency**, Springer, Switzerland 2019, p. 556 [555–574], https://doi.org/10.1007/978-3-030-25365-3 [emphasis in the original].

[22] See Enrico Cini and Vicenzo Fano, "Careful With Those Scissors, Eugene! Against the Observational Indistinguishability of Spacetimes", *Studies in History and Philosophy of Science Part A* 2021, Vol. 89, pp. 103–113, https://doi.org/10.1016/j.shpsa.2021.07.007.

Even beyond historical considerations, the Archytas paradox has “strained” the intuitions of some contemporary philosophers. Nerlich, for instance, states very significantly that:

> The thought that space could just come to an end is one at which the mind rebels. Let us express our distaste for such spaces by calling them pathological [...]. Why might one think that space cannot have boundaries? We cannot envisage any kind of mechanics for a world at the point at which moving objects just run out of places to go. What would it be like to push or throw such an object? We can’t envisage. [23]

What seems unimaginable to Nerlich here, as seen earlier, is one of the well-established properties of my Archytas paradox model. It fully explains the geometrical origin of the forces preventing an object from being pushed or thrown beyond the boundaries of space, be it (i)Lineland^0 (section 3.1), (i)Flatland^0 (section 3.2) or (i)Spaceland^0 (section 3.3). More cautiously, and closely related to the above, Sorabji argues that we should not rule out the possibility that there are spaces that have no spatial relationship to our own or to each other. [24] While giving no details as to how this possibility might be materialised, the Archytas paradox model seen above does so. The key is in the genesis of a manifold with boundary such as (i)Spaceland^0, in Figure 4. The points in (i)Spaceland^0 have no spatial relationship to the points in (i)Spaceland$^\pm$ as they are at an infinite distance from the latter.

5. The Archytas Paradox Model (APM) in context

The geometrical three-sphere is the set of points in the ordinary four-dimensional space that are at a given fixed distance from a given point (which is the three-dimensional surface of a four-dimensional ball; Einstein’s initial model of space). It is a finite and unbounded space, but with non-zero curvature. The simplest example of finite and unbounded space whose geometry is Euclidean is the so-called three-torus, [25] obtained by simply identifying the points on the opposite sides of a cube. However, the APM introduces a different model of space:

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[23] Graham Nerlich, **The Shape of Space**, Cambridge University Press, Cambridge 2009, pp. 56–57.

[24] See Richard Sorabji, **Matter, Space and Motion: Theories in Antiquity and Their Sequel**, Duckworth, London 1988.

the geometry of (i)Spaceland^0 is also Euclidean, but (i)Spaceland^0 itself is finite and *bounded*. It does, however, form part of a larger "superspace", (i)Spaceland, which also contains (i)Spaceland$^{\pm}$ and strictly models the Archytas paradox. Barring its bounded nature, (i)Spaceland^0 is a Euclidean space made up of all the points in (i)Spaceland that are at a finite distance from their spherical surface S_{LR} (or, equivalently, from its geometrical centre). Naturally, something with boundary, such as (i)Spaceland^0, cannot be a manifold. Clearly, the reason for this is that the points on S_{LR} (which, lest we forget, reside in (i)Spaceland) do not have neighborhoods which are diffeomorphic to tridimensional open balls. [26] This particularity would seem to cast some doubt on the interest and relevance of the APM. As follows from the previously mentioned indication made by Cini and Fano, our best spacetime physics considers predominately unbounded models and works with differentiable manifolds. Cao also recalls that any specification of boundary conditions is incompatible with Einstein's general theory of relativity as local physics. [27] However, this is not the whole story. Indeed, something with an edge, such as a disk, is not a manifold. To study such structures, the expression manifold-with-boundary has been introduced in the literature. And, as a matter of fact, manifolds with boundary have, for some years now, been a fundamental theoretical resource in a variety of approaches to quantum gravity and cosmology. [28] As I see it, this clearly justifies that the theoretical framework in which the APM is built (the genesis of a manifold with boundary) is neither extemporaneous nor irrelevant.

(i)Spaceland$^{\pm}$ has a "family resemblance" to a somewhat similar construction made by Poincaré in his well-known discussion on the conventionality of geometry. Poincaré argues that a particular non-Euclidean geometry would

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[25] See Jeffrey R. Weeks, **The Shape of Space**, CRC Press — Taylor & Francis Group, Boca Raton 2020.

[26] See Theodore Frankel, **The Geometry of Physics: An Introduction**, Cambridge University Press, Cambridge 2004.

[27] See Tian Yu Cao, **Conceptual Developments of 20th Century Field Theories**, Cambridge University Press, Cambridge 1997.

[28] For a pedagogical exposition in which they also review the physical and mathematical motivations behind studying physical theories in the presence of boundaries, see, for example: Giampiero Esposito, Alexander Yu. Kamenshchik, and Giuseppe Pollifrone, **Euclidean Quantum Gravity on Manifolds with Boundary**, Springer, Dordrecht 1997, https://doi.org/10.1007/978-94-011-5806-0.

prevent any observer from ever reaching a possible "edge" of the universe. [29] Indeed, this is what would also happen to an observer in (i)Spaceland$^{\pm}$ if they wished to reach the boundary S_{LR} of (i)Spaceland^{0}: no velocity that is bounded (no matter how large the boundary) will allow them to do so. Poincaré's space, however, is far from being adaptable so as to fulfil the same role as the APM. Indeed, a key feature of the APM is (i)Spaceland^{0}, which is finite, bounded and Euclidean, and where an observer can travel at any finite speed until they reach (in a finite time) the border S_{LR} of its space. In contrast, the Poincaré disk model is bounded and finite from an extrinsic viewpoint but unbounded and infinite from an intrinsic viewpoint.

A final word must be said regarding the origin of what is above called the Archytas force. This force manifests itself in certain regions of space due to the presence of purely spatial edges, not edges IN space. This presupposes (as stated on p. 8) that spatial points are not in space (rather they constitute it), which is consistent with a characteristic axiom in the mereological theory of location, the axiom of Conditional Emptiness ("regions do not have a location — they *are* locations" [30]). However, in the theory of spatial representation there is an alternative called Conditional Reflexivity ("spatial regions are themselves entities located [...] at themselves" [31]), which is incompatible with Conditional Emptiness. An advantage of the former is that it allows regionhood to be defined very simply from a primitive binary relation of location: *x* is a region if and only if it is the location of something. A disadvantage (in my view) is that, as Gilmore recalls, it proves incompatible with the plausible principle that two entities cannot share the same location. [32] In any event, Varzi also argues that there is no deep metaphysical issue behind these two options, and that they are equally good stipulations. However, when taken from a physical rather than metaphysical

[29] See an elementary description of Poincaré's space in: Huggett, **Everywhere and Everywhen...**, pp. 34–36.

[30] Achille C. Varzi, "Spatial Reasoning and Ontology: Parts, Whole, and Locations", in: Marco Aiello, Ian E. Pratt-Hartman, and Johan F.A.K. van Benthem (eds.), **Handbook of Spatial Logics**, Springer, Dordrecht 2007, p. 1016 [945–1038], https://doi.org/10.1007/978-1-4020-5587-4 [emphasis in the original].

[31] Varzi, "Spatial Reasoning and Ontology...", p. 1015.

[32] See Cody Gilmore, Claudio Calosi, and Damiano Costa, "Location and Mereology", in: Edward N. Zalta and Uri Nodelman (eds.), *The Stanford Encyclopedia of Philosophy*, Spring 2024 Edition, https://tiny.pl/q2h0mb_4 [14.04.2025].

perspective, from the spacetime perspective, the tetra-dimensional analogue of Conditional Emptiness seems to be the preferred option for philosophers of science. As Curiel argues when discussing spacetime singularities, a black hole is not a thing *in* spacetime; it is instead a feature of spacetime itself. [33] Similarly, the Archytas force has its origin in edges that are not a thing *in* space, rather a feature of space.

In conclusion, it should be noted that my Archytas Paradox Model (APM) has all the essential features of what is understood in standard philosophy of science as a model. Nevertheless, one does not usually model "in a void", but under the assumptions of some underlying theory. In the present case, the underlying theory can comfortably be considered to be Newtonian mechanics (rather than relativistic mechanics), and what is proposed is thus a Newtonian model of the Archytas paradox. This model constitutes (barring inconsistency) a conceptual possibility and, in this context, the term "physical possibility" means consistency with the Newtonian model of the Archytas paradox. The term "physically possible" is sometimes used in the strong (absolute) sense of "compatible with the laws of nature". This is, however, a meaning of the term that is of little use in philosophical discussion. Since we do not really know (beyond conjecture) the true laws of nature, neither do we know exactly what is possible in this strong sense. Compatibility with a model (or theory) entails a relativisation of the concept of physical possibility that makes it much more operational. Assuming such relativisation, the Newtonian nature (in a broad sense) of the APM enables the origin of the Archytas force at an edge to be fully clarified because, as already seen, a particle arriving at this edge must necessarily change its state of motion (and this, in Newtonian terms, precisely entails the presence of a force). [34]

Jon Pérez Laraudogoitia

[33] Erik Curiel, "Singularities and Black Holes", in: Edward N. Zalta and Uri Nodelman (eds.), *The Stanford Encyclopedia of Philosophy*, Summer 2023 Edition, https://tiny.pl/yxr9rnfv [14.04.2025].

[34] Note that force *F* exerted on the particle does not entail any reaction force — *F* exerted by the particle itself. Newton's third law is not satisfied, and for this reason the Newtonian nature of the APM is referred to "in the broad sense".

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